

On the intersection of the \mathcal{F} -maximal subgroups and the generalized \mathcal{F} -hypercentre of a finite group

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Abstract

Let \mathcal{F} be a class of groups. A chief factor H/K of a group G is called \mathcal{F} -central in G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathcal{F}$. We write $Z_{\pi\mathcal{F}}(G)$ to denote the product of all normal subgroups of G whose G -chief factors of order divisible by at least one prime in π are \mathcal{F} -central. We call $Z_{\pi\mathcal{F}}(G)$ the $\pi\mathcal{F}$ -hypercentre of G . A subgroup U of a group G is called \mathcal{F} -maximal in G provided that (a) $U \in \mathcal{F}$, and (b) if $U \leq V \leq G$ and $V \in \mathcal{F}$, then $U = V$. In this paper we study the properties of the intersection of all \mathcal{F} -maximal subgroups of a finite group. In particular, we analyze the condition under which $Z_{\pi\mathcal{F}}(G)$ coincides with the intersection of all \mathcal{F} -maximal subgroups of G .

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover p is always supposed to be a prime and π is a non-empty subset of the set \mathbb{P} of all primes. We use \mathcal{G}_π (\mathcal{S}_π) to denote the class of all π -groups (of all soluble π -groups, respectively). In particular, \mathcal{G}_p denotes the class of all p -groups; and we put that $\mathcal{G}_\emptyset = \mathcal{S}_\emptyset = (1)$ is the class of all identity groups. We also use \mathcal{N} , \mathcal{U} and \mathcal{S} to denote the classes of all nilpotent groups, of all supersoluble groups and of all soluble groups, respectively.

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Let \mathcal{F} be a class of groups. A group G is said to be \mathcal{F} -critical if G is not in \mathcal{F} but all proper subgroups of G are in \mathcal{F} [4, p. 517]. If $1 \in \mathcal{F}$, then we write $G^{\mathcal{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathcal{F}$. For any two classes \mathcal{F} and \mathcal{X} of groups, $\mathcal{X}\mathcal{F}$ is the class of groups G such that $G^{\mathcal{F}} \in \mathcal{X}$.

A *formation* is a class \mathcal{F} of groups with the following properties: (i) Every homomorphic image of any group in \mathcal{F} belongs to \mathcal{F} ; (ii) If $\mathcal{F} \neq \emptyset$, then $G/G^{\mathcal{F}} \in \mathcal{F}$ for any group G . A formation \mathcal{F} is said to be: *saturated* if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$; *hereditary* if $H \in \mathcal{F}$ whenever $H \leq G \in \mathcal{F}$.

For any formation function $f : \mathbb{P} \rightarrow \{\text{formations of groups}\}$, the symbol $LF(f)$ denotes the collection of all groups G such that either $G = 1$ or $G \neq 1$ and $G/C_G(H/K) \in f(p)$ for every chief factor H/K of G and every $p \in \pi(H/K)$. If $\mathcal{F} = LF(f)$ for some formation function f , then f is said to be a local definition or a *local satellite* (Shemetkov) of \mathcal{F} . Every non-empty saturated formation \mathcal{F} has a unique local satellite F with the following property: For any prime p , both $F(p) \subseteq \mathcal{F}$ and $G \in F(p)$ whenever $G/O_p(G) \in F(p)$ (see [4, IV, Proposition 3.8]). Such a satellite is called the *canonical local satellite* of \mathcal{F} .

A chief factor H/K of a group G is called \mathcal{F} -central in G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathcal{F}$. A normal subgroup N of G is said to be $\pi\mathcal{F}$ -hypercentral in G if either $N = 1$ or $N \neq 1$ and every chief factor of G below N of order divisible by at least one prime in π is \mathcal{F} -central in G . The symbol $Z_{\pi\mathcal{F}}(G)$ denotes the $\pi\mathcal{F}$ -hypercentre of G , that is, the product of all normal $\pi\mathcal{F}$ -hypercentral subgroups of G . It is clear that for the \mathcal{F} -hypercentre $Z_{\mathcal{F}}(G)$ of G (see [4, p. 389]) we have $Z_{\mathcal{F}}(G) = Z_{\mathbb{P}\mathcal{F}}(G)$. On the other limited case, when $\pi = \{p\}$, $Z_{p\mathcal{F}}(G)$ is the the product of all normal subgroups N of G such that every chief factor of G below N of order divisible by p is \mathcal{F} -central.

A subgroup U of G is called \mathcal{F} -maximal in G provided that (a) $U \in \mathcal{F}$, and (b) if $U \leq V \leq G$ and $V \in \mathcal{F}$, then $U = V$ [4, p. 288]. We denote the intersection of all \mathcal{F} -maximal subgroups of G by $\text{Int}_{\mathcal{F}}(G)$. In the paper [3], Beidleman and Heineken characterized the subgroup $\text{Int}_{\mathcal{F}}(G)$ in the case when G is soluble and \mathcal{F} is a hereditary saturated formation. In this paper, as a development of some results in [12] and [13], we find some new properties and applications of the subgroup $\text{Int}_{\mathcal{F}}(G)$.

Baer [1] proved that $\text{Int}_{\mathcal{N}}(G)$ coincides with the hypercentre $Z_{\infty}(G) = Z_{\mathcal{N}}(G)$ of G . Later, in [9], Sidorov proved that if \mathcal{F} the class of all soluble groups G of nilpotent length $l(G) \leq r$ ($r \in \mathbb{N}$), then for each soluble group G , the equality $Z_{\mathcal{F}}(G) = \text{Int}_{\mathcal{F}}(G)$ holds. In the papers [12] and [13], the analogous results were obtained for the classes of all p -decomposable groups and of all groups G with $G' \leq F(G)$ in the universe of all groups. As one of our results in this paper, we shall also prove that the intersection of all maximal p -nilpotent subgroups of G coincides with the subgroup $Z_{p\mathcal{N}}(G)$. But in general, $Z_{\pi\mathcal{F}}(G) \neq \text{Int}_{\mathcal{F}}(G)$, even when \mathcal{F} is the class of all supersoluble (all p -supersoluble, for any odd prime p) groups and G is soluble (see Theorem A and Remark 4.8 in Section 4).

Definition 1.1. Let \mathcal{X} be a non-empty class of groups and $\mathcal{F} = LF(F)$ be a saturated formation, where F is the canonical local satellite of \mathcal{F} . We say that \mathcal{F} satisfies the π -boundary condition (the boundary condition if $\pi = \mathbb{P}$) in \mathcal{X} if $G \in \mathcal{F}$ whenever $G \in \mathcal{X}$ and G is an $F(p)$ -critical group for at

least one $p \in \pi$.

We say that \mathcal{F} satisfies the π -boundary condition if \mathcal{F} satisfies the π -boundary condition in the class of all groups.

If \mathcal{F} is a non-empty formation with $\pi(\mathcal{F}) = \emptyset$, then $\mathcal{F} = (1)$, and therefore for any group G we have $Z_{\mathcal{F}}(G) = 1 = \text{Int}_{\mathcal{F}}(G)$. In the other limited case, when $\mathcal{F} = \mathcal{G}$ is the class of all groups, we have $Z_{\mathcal{F}}(G) = G = \text{Int}_{\mathcal{F}}(G)$. Similarly, if $\mathcal{F} = \mathcal{S}$, then $Z_{\mathcal{F}}(G) = G = \text{Int}_{\mathcal{F}}(G)$ for every soluble group G .

For the general case, we shall prove the following.

Theorem A. *Let \mathcal{F} be a hereditary saturated formation with $(1) \neq \mathcal{F} \neq \mathcal{G}$. Let $\pi \subseteq \pi(\mathcal{F})$. Then the equality*

$$Z_{\pi\mathcal{F}}(G) = \text{Int}_{\mathcal{F}}(G)$$

holds for each group G if and only if $\mathcal{N} \subseteq \mathcal{F} = \mathcal{G}_{\pi'}\mathcal{F}$ and \mathcal{F} satisfies the π -boundary condition.

Note that $N(p) = \mathcal{G}_p$, where N is the canonical local satellite of \mathcal{N} . Hence every $N(p)$ -critical group has prime order. Therefore \mathcal{N} satisfies the boundary condition and so the above-mentioned Baer's result is a first corollary of Theorem A.

Theorem B. *Let \mathcal{F} be a hereditary saturated formation of soluble groups with $(1) \neq \mathcal{F} \neq \mathcal{S}$. Let $\pi \subseteq \pi(\mathcal{F})$. Then the equality*

$$Z_{\pi\mathcal{F}}(G) = \text{Int}_{\mathcal{F}}(G)$$

holds for each soluble group G if and only if $\mathcal{N} \subseteq \mathcal{F} = \mathcal{S}_{\pi'}\mathcal{F}$ and \mathcal{F} satisfies the boundary condition in the class of all soluble groups.

If for some classes \mathcal{F} and \mathcal{M} of groups we have $\mathcal{F} \subseteq \mathcal{M}$, then every \mathcal{F} -maximal subgroup of a group is contained in some its \mathcal{M} -maximal subgroup. Nevertheless, the following example shows that in general, $\text{Int}_{\mathcal{F}}(G) \not\subseteq \text{Int}_{\mathcal{M}}(G)$.

Example 1.2. Let $\mathcal{F} = \mathcal{U}$ and \mathcal{M} be the class of all p -supersoluble groups, where $p > 2$. Let q be a prime dividing $p - 1$ and $G = P \rtimes (Q \rtimes C)$, where C is a group of order p , Q is a simple $\mathbb{F}_q G$ -module which is faithful for C and P is a simple $\mathbb{F}_p G$ -module which is faithful for $Q \rtimes C$. Then, clearly, $P = \text{Int}_{\mathcal{F}}(G)$ and $\text{Int}_{\mathcal{M}}(G) = 1$.

This example is a motivation for the following our result.

Theorem C. *Let $\mathcal{F} \subseteq \mathcal{M} = LF(M)$ be hereditary saturated formations with $\pi \subseteq \pi(\mathcal{F})$, where M is the canonical local satellites of \mathcal{M} .*

(a) *Suppose that $\mathcal{N} \subseteq \mathcal{M} = \mathcal{G}_{\pi'}\mathcal{M}$ and \mathcal{F} satisfies the π -boundary condition in \mathcal{M} . Then the inclusion*

$$\text{Int}_{\mathcal{F}}(G) \leq \text{Int}_{\mathcal{M}}(G)$$

holds for each group G .

(b) *If every (soluble) $M(p)$ -critical group*

belongs to \mathcal{F} for every $p \in \pi$, then $\mathcal{N} \subseteq \mathcal{M}$ and

$$\text{Int}_{\mathcal{F}}(G) \leq Z_{\pi\mathcal{M}}(G)$$

for every (soluble) group G .

Recall that a subgroup H of a group G is said to be \mathcal{F} -subnormal (in the sense of Kegel [8]) or K - \mathcal{F} -subnormal in G (see p.236 in [2]) if either $H = G$ or there exists a chain of subgroups

$$H = H_0 < H_1 < \dots < H_t = G$$

such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i} \in \mathcal{F}$ for all $i = 1, \dots, t$.

For any group G , we write $\text{Int}_{\mathcal{F}}^*(G)$ to denote the intersection of all non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroups of G . The following theorem shows that for any hereditary saturated formation \mathcal{F} with $\mathcal{N} \subseteq \mathcal{F}$, the intersection of all non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroups of a group G coincides with $\text{Int}_{\mathcal{F}}(G)$.

Theorem D. *Let \mathcal{F} be a hereditary saturated formation containing all nilpotent groups. Then the equality*

$$\text{Int}_{\mathcal{F}}^*(G) = \text{Int}_{\mathcal{F}}(G)$$

holds for each group G .

We prove Theorems A, B, C and D in Section 3. In Section 4 we give some examples and discuss applications of these theorems.

All unexplained notation and terminology are standard. The reader is referred to [4], [2] and [6] if necessary.

2 Preliminaries

In view of Proposition 3.16 in [4, IV], we have

Lemma 2.1. *Let $\mathcal{F} = LF(F)$ be a hereditary saturated formation, where F is the canonical local satellite of \mathcal{F} . Then for any prime p , the formation $F(p)$ is hereditary.*

We shall need in our proofs a few facts about the $\pi\mathcal{F}$ -hypercentre.

Lemma 2.2. *Let $\mathcal{F} = LF(F)$ be a saturated formation, where F is the canonical local satellite of \mathcal{F} . Let $\pi \subseteq \pi(\mathcal{F})$ and $\sigma = \pi(\mathcal{F}) \setminus \pi$. Let N and T be normal subgroups of G , and $A \leq G$.*

(1) *A chief factor H/K of G is \mathcal{F} -central if and only if $G/C_G(H/K) \in F(p)$ for all primes p dividing $|H/K|$.*

(2) *Every G -chief factor of $Z_{\pi\mathcal{F}}(G)$ of order divisible by at least one prime in π is \mathcal{F} -central.*

(3) *$Z_{\pi\mathcal{F}}(G)N/N \leq Z_{\pi\mathcal{F}}(G/N)$.*

(4) *$Z_{\pi\mathcal{F}}(A)N/N \leq Z_{\pi\mathcal{F}}(AN/N)$.*

(5) If \mathcal{F} is (normally) hereditary and H is a (normal) subgroup of G , then $Z_{\pi\mathcal{F}}(H) \cap E \leq Z_{\pi\mathcal{F}}(H \cap E)$.

(6) If $\mathcal{G}_\sigma\mathcal{F} = \mathcal{F}$ and $G/Z_{\pi\mathcal{F}}(G) \in \mathcal{F}$, then $G \in \mathcal{F}$.

(7) Suppose that \mathcal{F} is (normally) hereditary and let H be a (normal) subgroup of G . If $\mathcal{G}_\sigma\mathcal{F} = \mathcal{F}$ and $H \in \mathcal{F}$, then $Z_{\pi\mathcal{F}}(G)H \in \mathcal{F}$.

Proof. (1) This assertion is well-known (see for example Theorem 17.14 in [?] or Theorem 3.1.6 in [6]). Assertions (2) and (6) are evident.

(3) Let H/K be a chief factor of G such that $N \leq K < H \leq NZ_{\pi\mathcal{F}}(G)$ and $|H/K|$ is divisible by at least one prime in π . Then H/K is G -isomorphic to the chief factor $H \cap Z_{\pi\mathcal{F}}(G)/K \cap Z_{\pi\mathcal{F}}(G)$ of G . Therefore H/K is \mathcal{F} -central in G by (1) and (2). Consequently, $Z_{\pi\mathcal{F}}(G)N/N \leq Z_{\pi\mathcal{F}}(G/N)$.

(4) Let $f : A/A \cap N \rightarrow AN/N$ be the canonical isomorphism from $A/A \cap N$ onto AN/N . Then $f(Z_{\pi\mathcal{F}}(A/A \cap N)) = Z_{\pi\mathcal{F}}(AN/N)$ and

$$f(Z_{\pi\mathcal{F}}(A)(A \cap N)/(A \cap N)) = Z_{\pi\mathcal{F}}(A)N/N.$$

By (3) we have

$$Z_{\pi\mathcal{F}}(A)(A \cap N)/(A \cap N) \leq Z_{\pi\mathcal{F}}(A/A \cap N).$$

Hence $Z_{\pi\mathcal{F}}(A)N/N \leq Z_{\pi\mathcal{F}}(AN/N)$.

(5) First suppose that \mathcal{F} is hereditary. Let

$$1 = Z_0 < Z_1 < \dots < Z_t = Z_{\pi\mathcal{F}}(G)$$

be a chief series of G below $Z_{\pi\mathcal{F}}(G)$ and $C_i = C_G(Z_i/Z_{i-1})$. Let q be a prime divisor of

$$|Z_i \cap H/Z_{i-1} \cap H| = |Z_{i-1}(Z_i \cap H)/Z_{i-1}|.$$

Suppose that q divides $|Z_i \cap H/Z_{i-1} \cap H|$. Then q divide $|Z_i/Z_{i-1}|$, so $G/C_i \in F(q)$ by (1). Hence $H/H \cap C_i \simeq C_i H/C_i \in F(q)$. But $H \cap C_i \leq C_H(Z_i \cap H/Z_{i-1} \cap H)$. Hence $H/C_H(Z_i \cap H/Z_{i-1} \cap H) \in F(q)$ for all primes q dividing $|Z_i \cap H/Z_{i-1} \cap H|$. Thus $Z_{\pi\mathcal{F}}(G) \cap H \leq Z_{\pi\mathcal{F}}(H)$ by (1) and (2). But then

$$Z_{\pi\mathcal{F}}(H) \cap E = Z_{\pi\mathcal{F}}(H) \cap (H \cap E) \leq Z_{\pi\mathcal{F}}(H \cap E).$$

Similarly, one may prove the second assertion of (5).

(7) Since $H \in \mathcal{F}$ we have

$$HZ_{\pi\mathcal{F}}(G)/Z_{\pi\mathcal{F}}(G) \simeq H/H \cap Z_{\pi\mathcal{F}}(G) \in \mathcal{F}$$

and

$$Z_{\pi\mathcal{F}}(G) \leq Z_{\pi\mathcal{F}}(Z_{\pi\mathcal{F}}(G)H)$$

by (5). Hence $HZ_{\pi\mathcal{F}}(G) \in \mathcal{F}$ by (6).

The lemma is proved.

The following lemma is evident (Note only that Statement (i) directly follows from [4, Theorem A.9.2(c), p. 30]).

Lemma 2.3. *Let \mathcal{F} be a hereditary saturated formation. Let $N \leq U \leq G$, where N is a normal subgroup of G .*

- (i) *If $G/N \in \mathcal{F}$ and V is a minimal supplement of N in G , then $V \in \mathcal{F}$.*
- (ii) *If U/N is an \mathcal{F} -maximal subgroup of G/N , then $U = U_0N$ for some \mathcal{F} -maximal subgroup U_0 of G .*
- (iii) *If V is an \mathcal{F} -maximal subgroup of U , then $V = H \cap U$ for some \mathcal{F} -maximal subgroup H of G .*

The proofs of our theorems are based on the following general facts on the subgroup $\text{Int}_{\mathcal{F}}(G)$.

Lemma 2.4. *Let \mathcal{F} be a hereditary saturated formation, $\pi \subseteq \pi(\mathcal{F})$ and $\sigma = \pi(\mathcal{F}) \setminus \pi$. Let H, E be subgroups of G , N a normal subgroup of G and $I = \text{Int}_{\mathcal{F}}(G)$. Then:*

- (a) $\text{Int}_{\mathcal{F}}(H)N/N \leq \text{Int}_{\mathcal{F}}(HN/N)$.
- (b) $\text{Int}_{\mathcal{F}}(H) \cap E \leq \text{Int}_{\mathcal{F}}(H \cap E)$.
- (c) *If $H/H \cap I \in \mathcal{F}$, then $H \in \mathcal{F}$.*
- (d) *If $H \in \mathcal{F}$, then $IH \in \mathcal{F}$.*
- (e) *If $N \leq I$, then $I/N = \text{Int}_{\mathcal{F}}(G/N)$.*
- (f) $\text{Int}_{\mathcal{F}}(G/I) = 1$.
- (g) *If $\mathcal{G}_{\sigma}\mathcal{F} = \mathcal{F}$, then $Z_{\pi\mathcal{F}}(G) \leq I$.*

Proof. Assertions (a)-(f) are proved in [12]. Now we prove (g). Let H be a subgroup of G such that $H \in \mathcal{F}$. Then $HZ_{\pi\mathcal{F}}(G)/Z_{\pi\mathcal{F}}(G) \simeq H/H \cap Z_{\pi\mathcal{F}}(G) \in \mathcal{F}$ and $Z_{\pi\mathcal{F}}(G) \leq Z_{\pi\mathcal{F}}(HZ_{\pi\mathcal{F}}(G))$ by Lemma 2.2(5). It follows from Lemma 2.2(5) that $HZ_{\pi\mathcal{F}}(G) \in \mathcal{F}$. Thus $Z_{\pi\mathcal{F}}(G) \leq I$.

The following lemma is a corollary of general results on f -hypercentral action (see [4, Chapter IV, Section 6]). For reader's convenience, we give a direct proof.

Lemma 2.5. *Let $\mathcal{F} = LF(F)$ be a saturated formation, where F is the canonical local satellite of \mathcal{F} . Let E be a normal p -subgroup of G . If $E \leq Z_{\mathcal{F}}(G)$, then $G/C_G(E) \in F(p)$.*

Proof. Let $1 = E_0 < E_1 < \dots < E_t = E$ be a chief series of G below E . Let $C_i = C_G(E_i/E_{i-1})$ and $C = C_1 \cap \dots \cap C_t$. Then $C_G(E) \leq C$ and so $C/C_G(E)$ is a p -group by Corollary 3.3 in [5, Chapter 5]. On the other hand, by Lemma 2.2(1), $G/C_i \in F(p)$, so $G/C \in F(p)$. Hence $G/C_G(E) \in F(p) = \mathcal{G}_p F(p)$. The lemma is proved.

Lemma 2.6. *Let $\mathcal{F} = LF(F)$ and \mathcal{M} be saturated formations with $p \in \pi(\mathcal{F})$ and $\mathcal{F} \subseteq \mathcal{M}$, where F is the canonical local satellite of \mathcal{F} . Suppose that G is a group of minimal order in the set of all $F(p)$ -critical groups $G \in \mathcal{M}$ with $G \notin \mathcal{F}$. Then $O_p(G) = 1 = \Phi(G)$ and $G^{\mathcal{F}}$ is the unique minimal*

normal subgroup of G .

Proof. Let N be a minimal normal subgroup of G . First we show that $G/N \in \mathcal{F} \cap \mathcal{M}$. Indeed, since $G \in \mathcal{M}$ and \mathcal{M} is a formation, $G/N \in \mathcal{M}$. Suppose that $G/N \notin \mathcal{F}$. Then $G/N \notin F(p)$ since $F(p) \subseteq \mathcal{F}$. On the other hand, for any maximal subgroup M/N of G/N we have $M/N \in F(p)$ since $F(p)$ is a formation and G is an $F(p)$ -critical group. Thus G/N is an $F(p)$ -critical group in with $G/N \notin \mathcal{M} \setminus \mathcal{F}$, which contradicts the minimality of G . Hence $G/N \in \mathcal{F}$. Since \mathcal{F} is a saturated formation, $N = G^{\mathcal{F}}$ is a unique minimal normal subgroup of G and $\Phi(G) = 1$. Suppose that $N \leq O_p(G)$ and let M be a maximal subgroup of G such that $G = NM$. Then $G/N \simeq M/N \cap M \in F(p) = \mathcal{G}_p F(p)$, so $G \leq F(p) \subseteq \mathcal{F}$. This contradiction shows that $O_p(G) = 1$. The lemma is proved.

Lemma 2.7. Let \mathcal{F} be a formation, H and E be subgroups of a group G , where H is K - \mathcal{F} -subnormal in G . Then:

- (i) $H \cap E$ is K - \mathcal{F} -subnormal in E (see Theorem 6.1.7 in [2]).
- (ii) If E is normal in G , then HE/E is K - \mathcal{F} -subnormal in G/E (see Theorem 6.1.6 in [2]).

Lemma 2.8. Let \mathcal{F} be a hereditary saturated formation. Let $N \leq U \leq G$, where N is a normal subgroup of G .

- (i) If U/N is a non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup of G/N , then $U = U_0N$ for some non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup U_0 of G .
- (ii) If V is a non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup of U , then $V = H \cap U$ for some non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup H of G .

Proof. (i) By Lemma 2.3 (ii), there is an \mathcal{F} -maximal subgroup U_0 of G such that $U = U_0N$. Since U/N is non- K - \mathcal{F} -subnormal in G/N , U_0 is not K - \mathcal{F} -subnormal in G by Lemma 2.7(ii).

(ii) By Lemma 2.3(iii), for some \mathcal{F} -maximal subgroup H of G we have $V = H \cap U$. Since V is a non- K - \mathcal{F} -subnormal in U , H is not K - \mathcal{F} -subnormal in G by Lemma 2.7(i).

Lemma 2.9. Let $\mathcal{F} = LF(F)$ be a non-empty saturated formation, where F is the canonical local satellite of \mathcal{F} .

- (1) If $\mathcal{F} = \mathcal{G}_p \mathcal{F}$ for some prime p , then $F(p) = \mathcal{F}$.
- (2) If $\mathcal{F} = \mathcal{NH}$ for some non-empty formation \mathcal{H} , then $F(p) = \mathcal{G}_p \mathcal{H}$ for all primes p .

Proof. (1) Since $F(p) \subseteq \mathcal{F}$, we need only prove that $\mathcal{F} \subseteq F(p)$. Suppose that this is false and let A be a group of minimal order in $\mathcal{F} \setminus F(p)$. Then $A^{F(p)}$ is a unique minimal normal subgroup of A since $F(p)$ is a formation. Moreover, $O_p(A) = 1$ since $F(p) = \mathcal{G}_p F(p)$. Let $G = C_p \wr A = K \rtimes A$ where K is the base group of the regular wreath G . Then $K = O_{p',p}(G)$ and $G \in \mathcal{F} = \mathcal{G}_p \mathcal{F}$. Hence $A \simeq G/P = G/O_{p',p}(G) \in F(p)$, a contradiction. Thus $F(p) = \mathcal{F}$.

(2) The inclusion $F(p) \subseteq \mathcal{G}_p \mathcal{H}$ is evident. The inverse inclusion can be proved similarly as the inclusion $\mathcal{F} \subseteq F(p)$ in the proof of (1).

We will also use in our proofs the following well-known elementary fact (see for example, [?,

Lemma 18.8] or [6, Lemma 3.5.13]).

Lemma 2.10. *If $O_p(G) = 1$ and G has a unique minimal normal subgroup, then there exists a simple $\mathbb{F}_p G$ -module which is faithful for G .*

3 Proofs of the Theorems

Proof of Theorem C. (a) Suppose that this assertion is false and let G be a counterexample of minimal order. Let $I = \text{Int}_{\mathcal{F}}(G)$ and $I_1 = \text{Int}_{\mathcal{M}}(G)$. Then $1 < I < G$ and $I_1 \neq G$. Let L be a minimal normal subgroup of G contained in I and $C = C_G(L)$. Then $\pi(L) \subseteq \pi(\mathcal{F})$.

(1) $IN/N \leq \text{Int}_{\mathcal{F}}(G/N) \leq \text{Int}_{\mathcal{M}}(G/N)$ for any non-identity normal subgroup N of G .

Indeed, by Lemma 2.4(a), we have $IN/N \leq \text{Int}_{\mathcal{F}}(G/N)$. On the other hand, by the choice of G , $\text{Int}_{\mathcal{F}}(G/N) \leq \text{Int}_{\mathcal{M}}(G/N)$.

(2) $L \not\leq I_1$; in particular, the order of L is divisible by some prime $p \in \pi$.

Suppose that $L \leq I_1$. Then $I_1/L = \text{Int}_{\mathcal{M}}(G/L)$ by Lemma 2.4(e). But by (1), $IL/L = I/L \leq \text{Int}_{\mathcal{F}}(G/L) \leq \text{Int}_{\mathcal{M}}(G/L)$. Hence $I/L \leq I_1/L$ and so $I \leq I_1$, a contradiction. Thus $L \not\leq I_1$. This means that there exists an \mathcal{M} -maximal subgroup M of G such that $L \not\leq M$. Suppose that L is a π' -group. Then $LM \in \mathcal{G}_{\pi'}\mathcal{M} = \mathcal{M}$, which contradicts the maximality of M . Hence the order of L is divisible by some prime $p \in \pi$.

(3) If $L \leq M < G$, then $L \leq \text{Int}_{\mathcal{M}}(M)$.

By Lemma 2.4(b), $L \leq I \cap M \leq \text{Int}_{\mathcal{F}}(M)$. But since $|M| < |G|$, $\text{Int}_{\mathcal{F}}(M) \leq \text{Int}_{\mathcal{M}}(M)$ by the choice of G . Hence $L \leq \text{Int}_{\mathcal{M}}(M)$.

(4) $G = LU$ for any \mathcal{M} -maximal subgroup U of G not containing L . In particular, $G/L \in \mathcal{M}$.

Indeed, suppose that $LU \neq G$. Then by (3), $L \leq \text{Int}_{\mathcal{M}}(LU)$, which implies that $LU \in \mathcal{M}$ by Lemma 2.4(c). This contradicts the \mathcal{M} -maximality of U . Hence we have (4).

(5) $C_G(L) \cap U = U_G = 1$ for any \mathcal{M} -maximal subgroup U of G not containing L .

Since $C_G(L)$ is normal in G and $G = LU$ by (4), $U_G = C_G(L) \cap U$. Assume that $U_G \neq 1$. Let $U/U_G \leq W/U_G$, where W/U_G is an \mathcal{M} -maximal subgroup of G/U_G . Then by (1), $LU_G/U_G \leq W/U_G$. Hence $G = LU \leq W$ by (4), which means that $G/U_G = W/U_G \in \mathcal{M}$. But by (4), $G/L \in \mathcal{M}$. Therefore $G \cong G/L \cap U_G \in \mathcal{M}$, and consequently $I = G$, a contradiction. Hence (5) holds.

The final contradiction for (a).

Since $L \not\leq I_1$ by (2), there is an \mathcal{M} -maximal subgroup M of G such that $L \not\leq M$. But then $G = LM$ by (4). Since $L \leq I$ and $G \notin \mathcal{F}$, $M \notin \mathcal{F}$ by Lemma 2.4(d). Let H be an \mathcal{F} -critical subgroup of M , V a maximal subgroup of H . We show that $V \in F(p)$. By Lemma 2.4(d), $D = LV \in \mathcal{F}$. Hence $D/O_{p',p}(D) \in F(p)$. First assume that L is a non-abelian group. Then, since p divides $|L|$, $O_{p',p}(D) \cap L = 1$. Hence $O_{p',p}(D) \leq C_G(L)$ and $O_{p',p}(D) \cap V = 1$ by (5). Since \mathcal{F}

is hereditary, $F(p)$ is hereditary by Lemma 2.1. Therefore $O_{p',p}(D)V/O_{p',p}(D) \simeq V \in F(p)$. Now assume that L is an abelian p -group. Then $L \leq O_{p',p}(D)$ and $O_{p',p}(D) = L(O_{p',p}(D) \cap V)$. Hence $O_{p'}(D) \leq M \cap C_G(L) = 1$. It follows that $O_{p',p}(D) = O_p(D)$. Therefore $D/O_p(D) \in F(p) = \mathcal{G}_p F(p)$, which implies that $D \in F(p)$ and so $V \in F(p)$. Therefore H is an $F(p)$ -critical group. Since \mathcal{M} is hereditary and $M \in \mathcal{M}$, $H \in \mathcal{M}$. But then $H \in \mathcal{F}$ since \mathcal{F} satisfies the π -boundary condition in \mathcal{M} by hypothesis. This contradiction completes the proof of (a).

(b) Suppose that every $M(p)$ -critical group G belongs to \mathcal{F} for every $p \in \pi$. First we show that $\mathcal{N} \subseteq \mathcal{M}$. Assume that this is false and let C_q be a group of prime order q with $C_q \notin \mathcal{M}$. Let $p \in \pi$. Then C_q is $M(p)$ -critical and so $C_q \in \mathcal{F} \subseteq \mathcal{M}$ by the hypothesis. This contradiction shows that $\mathcal{N} \subseteq \mathcal{M}$.

Now we show that $\text{Int}_{\mathcal{F}}(G) \leq Z_{\pi\mathcal{M}}(G)$ for every group G . Suppose that this assertion is false and let G be a counterexample of minimal order. Let $I = \text{Int}_{\mathcal{F}}(G)$ and $Z = Z_{\pi\mathcal{M}}(G)$. Then $1 < I < G$ and $Z \neq G$. Let N be a minimal normal subgroup of G and L a minimal normal subgroup of G contained in I . Then $\pi(L) \leq \pi(\mathcal{F})$. We proceed via the following steps.

(1) $IN/N \leq \text{Int}_{\mathcal{F}}(G/N) \leq Z_{\pi\mathcal{M}}(G/N)$.

Indeed, by Lemma 2.4(a), we have $IN/N \leq \text{Int}_{\mathcal{F}}(G/N)$. On the other hand, by the choice of G , $\text{Int}_{\mathcal{F}}(G/N) \leq Z_{\pi\mathcal{M}}(G/N)$.

(2) $L \not\leq Z$; in particular, the order of L is divisible by some prime $p \in \pi$.

Suppose that $L \leq Z$. Then, clearly, $Z/L = Z_{\pi\mathcal{M}}(G/L)$, and $I/L = \text{Int}_{\mathcal{F}}(G/L)$ by Lemma 2.4(e). But by (1), $\text{Int}_{\mathcal{F}}(G/L) \leq Z_{\pi\mathcal{M}}(G/L)$. Hence $I/L \leq Z/L$. Consequently, $I \leq Z$, a contradiction.

(3) If $L \leq M < G$, then $L \leq Z_{\mathcal{M}}(M)$.

By Lemma 2.4(b), $L \leq I \cap M \leq \text{Int}_{\mathcal{F}}(M)$. But since $|M| < |G|$, we have that $\text{Int}_{\mathcal{F}}(M) \leq Z_{\pi\mathcal{M}}(M)$ by the choice of G . Hence $L \leq Z_{\pi\mathcal{M}}(M)$ and so $L \leq Z_{\mathcal{M}}(M)$ since the order of L is divisible by some prime $p \in \pi$ by (2).

(4) $L = N$ is the unique minimal normal subgroup of G .

Suppose that $L \neq N$. Then by (1), $NL/N \leq Z_{\pi\mathcal{M}}(G/N)$. Hence from the G -isomorphism $NL/N \simeq L$ we obtain $L \leq Z$, which contradicts (2).

(5) $L \not\leq \Phi(G)$.

Suppose that $L \leq \Phi(G)$. Then L is a p -group by (2). Let $C = C_G(L)$ and M be any maximal subgroup of G . Then $L \leq M$. Hence $L \leq Z_{\mathcal{M}}(M)$ by (3), so $M/M \cap C \in M(p)$ by Lemmas 2.1(1) and Lemma 2.5. If $C \not\leq M$, then $G/C = CM/C \simeq M/M \cap C \in M(p)$. This implies that $L \leq Z$, which contradicts (2). Hence $C \leq M$ for all maximal subgroups M of G . It follows that C is nilpotent. Then in view of (4), C is a p -group since C is normal in G . Hence for every maximal subgroup M of G we have $M \in \mathcal{G}_p M(p) = M(p)$. But since $M(p) \subseteq \mathcal{M}$, $G \notin M(p)$ (otherwise $G \in \mathcal{M}$ and so $G = Z$). This shows that G is an $M(p)$ -critical group. Therefore $G \in \mathcal{F}$ by the hypothesis. But since

$\mathcal{F} \subseteq \mathcal{M}$, we have $G \in \mathcal{M}$ and so $G = Z$, a contradiction. Thus (5) holds.

(6) $C = C_G(L) \leq L$ (This follows from (4), (5) and Theorem 15.6 in [4, Chapter A]).

(7) If $L \leq M < G$, then $M \in M(p)$.

First by (3), $L \leq Z_{\mathcal{M}}(M)$. If $L = C$, then $M/L = M/M \cap C \in M(p)$ by Lemma 2.5, which implies that $M \in \mathcal{G}_p M(p) = M(p)$ since L is a p -group by (2). Now suppose that L is a non-abelian group. Let $1 = L_0 < L_1 < \dots < L_n = L$ be a chief series of M below L . Let $C_i = C_M(L_i/L_{i-1})$ and $C_0 = C_1 \cap \dots \cap C_n$. Since $L \leq Z_{\mathcal{M}}(M)$, $M/C_i \in M(p)$ for all $i = 1, \dots, n$. It follows that $M/C_0 \in M(p)$. Since $C = 1$ by (4) and (6), for any minimal normal subgroup R of M we have $R \leq L$. Suppose that $C_0 \neq 1$ and let R be a minimal normal subgroup of M contained in C_0 . Then $R \leq L$ and $R \leq C_M(H/K)$ for each chief factor H/K of M . Thus $R \leq F(M)$ is abelian and hence L is abelian. This contradiction shows that $C_0 = 1$. Consequently, $M \in M(p)$.

(8) There exists a subgroup U of G such that $U \in \mathcal{F}$ and $LU = G$.

Indeed, suppose that every maximal subgroup of G not containing L belongs to $M(p)$. Then by (7), G is an $M(p)$ -critical group. Hence $G \in \mathcal{F}$ by the hypothesis. But then $I = G$, a contradiction. Hence there exists a maximal subgroup M of G such that $G = LM$ and $M \notin M(p)$. Take an $M(p)$ -critical subgroup U of M . Then in view of (7), $LU = G$ and $U \in \mathcal{F}$ by the hypothesis.

(9) The final contradiction for (b).

Since $L \leq I$ and $G/L = UL/L \simeq U/U \cap L \in \mathcal{F}$ by (8), it follows from Lemma 2.4(c) that $G \in \mathcal{F}$ and so $G = I$. The final contradiction shows that $\text{Int}_{\mathcal{F}}(G) \leq Z_{\pi\mathcal{M}}(G)$ for every group G . The second assertion of (b) can be proved similarly. The theorem is proved.

Proofs of Theorems A and B. Since $Z_{\mathcal{F}}(G) \leq \text{Int}_{\mathcal{F}}(G)$ by Lemma 2.4(g), the sufficiency is a special case, when $\mathcal{F} = \mathcal{M}$, of Theorem C (b). Now suppose that the equality $Z_{\pi\mathcal{F}}(G) = \text{Int}_{\mathcal{F}}(G)$ holds for each (soluble) group G .

First we show that $\mathcal{N} \subseteq \mathcal{F}$. Let F be the canonical local satellite of \mathcal{F} . Suppose that for some group C_q of prime order q we have $C_q \notin \mathcal{F}$. Let $p \in \pi$ and $G = PC_q$, where P is a simple $\mathbb{F}_p A$ -module P which is faithful for C_q . Then $P = \text{Int}_{\mathcal{F}}(G)$ and $Z_{\mathcal{F}}(G) = 1$ since $F(p) \subseteq \mathcal{F}$. This contradiction shows that $\mathcal{N} \subseteq \mathcal{F}$.

Now we show that $\mathcal{G}_{\pi'}\mathcal{F} = \mathcal{F}$ ($\mathcal{S}_{\pi'}\mathcal{F} = \mathcal{F}$, respectively). The inclusion $\mathcal{F} \subseteq \mathcal{G}_{\pi'}\mathcal{F}$ is evident. Suppose that $\mathcal{G}_{\pi'}\mathcal{F} \not\subseteq \mathcal{F}$ ($\mathcal{S}_{\pi'}\mathcal{F} \not\subseteq \mathcal{F}$) and let G be a group of minimal order in $\mathcal{G}_{\pi'}\mathcal{F} \setminus \mathcal{F}$ (in $\mathcal{S}_{\pi'}\mathcal{F} \setminus \mathcal{F}$, respectively). Then $G^{\mathcal{F}}$ is the unique minimal normal subgroup of G and $G^{\mathcal{F}}$ is a π' -group. Hence

$$G^{\mathcal{F}} \leq Z_{\pi\mathcal{F}}(G) = \text{Int}_{\mathcal{F}}(G).$$

It follows from Lemma 2.4(c) that $G \in \mathcal{F}$. This contradiction shows that $\mathcal{G}_{\pi'}\mathcal{F} = \mathcal{F}$ ($\mathcal{S}_{\pi'}\mathcal{F} = \mathcal{F}$, respectively).

Finally, we show that \mathcal{F} satisfies the π -boundary condition (the π -boundary condition in the class \mathcal{S} , respectively). Suppose that this is false. Then for some $p \in \pi$, the set of all (soluble) $F(p)$ -critical

groups A with $A \notin \mathcal{F}$ is non-empty. Let us choose in this set a group A with minimal $|A|$. Then by Lemma 2.6, $A^{\mathcal{F}}$ is the unique minimal normal subgroup of G and $O_p(A) = 1 = \Phi(A)$. Hence by Lemma 2.10, there exists a simple $\mathbb{F}_p A$ -module P which is faithful for A . Let $G = P \rtimes A$ and M be any maximal subgroup of G . If $P \not\leq M$, then $M \simeq G/P \simeq A \notin \mathcal{F}$. On the other hand, if $P \leq M$, then $M = M \cap PA = P(M \cap A)$, where $M \cap A$ is a maximal subgroup of A . Hence $M \cap A \in F(p)$ and so $M \in \mathcal{G}_p F(p) = F(p) \subseteq \mathcal{F}$. Therefore P is contained in the intersection of all \mathcal{F} -maximal subgroups of G . Then $P \leq Z_{\pi\mathcal{F}}(G)$ by our assumption on \mathcal{F} . It follows that $A \simeq G/P = G/C_G(P) \in F(p) \subseteq \mathcal{F}$ by Lemmas 2.2(1) and Lemma 2.5. But this contradicts the choice of A . Therefore \mathcal{F} satisfies the π -boundary condition (\mathcal{F} satisfies the π -boundary condition in the class \mathcal{S}). The theorems are proved.

In view of Theorems A, B and C we have

Corollary 3.1. *Let \mathcal{F} be a hereditary saturated formation with $(1) \neq \mathcal{F} \neq \mathcal{G}$. Then the equality $\text{Int}_{\mathcal{F}}(G) = Z_{\mathcal{F}}(G)$ holds for each group G if and only if $\mathcal{N} \subseteq \mathcal{F}$ and \mathcal{F} satisfies the boundary condition.*

Corollary 3.2. *Let \mathcal{F} be a hereditary saturated formation of soluble groups with $(1) \neq \mathcal{F} \neq \mathcal{S}$. Then the equality $\text{Int}_{\mathcal{F}}(G) = Z_{\mathcal{F}}(G)$ holds for each soluble group G if and only if $\mathcal{N} \subseteq \mathcal{F}$, \mathcal{F} satisfies the boundary condition in the class \mathcal{S} .*

Note that Corollary 3.2 also follows directly from [3, Main Theorem].

Proof of Theorem D. We will prove the theorem by induction on $|G|$. If $G \in \mathcal{F}$, then

$$\text{Int}_{\mathcal{F}}^*(G) = G = \text{Int}_{\mathcal{F}}(G).$$

We may, therefore, assume that $G \notin \mathcal{F}$. Let $I = \text{Int}_{\mathcal{F}}(G)$, $I^* = \text{Int}_{\mathcal{F}}^*(G)$ and N a minimal normal subgroup of G . Then $I \leq I^*$. Hence we may assume that $I^* \neq 1$.

$$(1) \quad I^*N/N \leq \text{Int}_{\mathcal{F}}^*(G/N).$$

If U/N is a non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup of G/N , then for some non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup U_0 of G we have $U = U_0N$ by Lemma 2.8(i). Let

$$\text{Int}_{\mathcal{F}}^*(G/N) = U_1/N \cap \dots \cap U_t/N,$$

where U_i/N is a non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup of G/N for all $i = 1, \dots, t$. Let V_i be a non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup of G such that $U_i = V_iN$. Then $I^* \leq V_1 \cap \dots \cap V_t$. Hence $I^*N/N \leq \text{Int}_{\mathcal{F}}^*(G/N)$.

$$(2) \quad \text{If } N \leq I^*, \text{ then } \text{Int}_{\mathcal{F}}^*(G/N) = I^*/N.$$

By Lemma 2.8(i), it is enough to prove that if U is a non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup of G , then U/N is a non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup of G/N . Let $U/N \leq X/N$, where X/N is a non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup of G/N . By Lemma 2.8(i), $X = U_0N$ for some non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup U_0 of G . But since $N \leq U_0$, $U/N \leq U_0/N$ and so $U = U_0$. Thus $U/N = X/N$.

$$(3) \quad \text{If } I^* \cap H \leq \text{Int}_{\mathcal{F}}^*(H) \text{ for any subgroup } H \text{ of } G.$$

Let V be an arbitrary non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup of H . Then $V = H \cap U$ for some non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroup U of G by Lemma 2.8(ii). Thus there are non- K - \mathcal{F} -subnormal \mathcal{F} -maximal subgroups U_1, \dots, U_t of G such that

$$\text{Int}_{\mathcal{F}}^*(H) = U_1 \cap \dots \cap U_t \cap H.$$

This induces that $I^* \cap H \leq \text{Int}_{\mathcal{F}}^*(H)$.

(4) If $E = RV$ for some normal subgroup R of G contained in I^* and K - \mathcal{F} -subnormal subgroup $V \in \mathcal{F}$, then $E \in \mathcal{F}$.

First note that by (3), $R \leq \text{Int}_{\mathcal{F}}^*(E)$. On the other hand, by Lemma 2.7(i), V is a K - \mathcal{F} -subnormal subgroup of E . Hence we need only consider the case when $G = E$. Assume that $G \notin \mathcal{F}$. Then $D \neq 1$. Let R be any minimal normal subgroup of G . Then $(DR/R)(VR/R) = G/R$, where $DR/R \leq \text{Int}_{\mathcal{F}}^*(G/R)$ by (2), and $VR/R \simeq V/V \cap R \in \mathcal{F}$. On the other hand, by induction we have $\text{Int}_{\mathcal{F}}^*(G/R) = \text{Int}_{\mathcal{F}}(G/R)$, so $G/R \in \mathcal{F}$ by Lemma 2.4 (d). This implies that R is the only minimal normal subgroup of G and so $R = G^{\mathcal{F}} \leq \text{Int}_{\mathcal{F}}^*(G)$. Let W be a minimal supplement of R in G . Then $W \in \mathcal{F}$ by Lemma 2.3(i). Let $W \leq E$, where E is an \mathcal{F} -maximal subgroup of G . If E is not K - \mathcal{F} -subnormal in G , then $R \leq E$. Thus $G = RW = RE = E \in \mathcal{F}$, a contradiction. This shows that E is K - \mathcal{F} -subnormal in G . But then there is a proper subgroup X of G such that $E \leq X$ and either X is normal in G or $R = G^{\mathcal{F}} \leq X$. In both of this cases, we have that $G = RV = RX = X < G$, a contradiction. Hence we have (4).

Conclusion.

Let R be a minimal normal subgroup of G contained in I^* . If $R \leq I$, then $I/R = \text{Int}_{\mathcal{F}}(G/R)$ by Lemma 2.3(e), and $I^*/R = \text{Int}_{\mathcal{F}}^*(G/R)$ by (2). Therefore by induction, $\text{Int}_{\mathcal{F}}^*(G/R) = \text{Int}_{\mathcal{F}}(G/R)$. It follows that $I = I^*$.

Finally, suppose that $R \not\leq I$. Then $R \not\leq U$ for some \mathcal{F} -maximal subgroup U of G . It is clear that U is a K - \mathcal{F} -subnormal subgroup of G and hence $E \in \mathcal{F}$ by (4). But then $E = U$, which implies $R \leq U$, a contradiction. The theorem is proved.

4 Applications and Remarks

Applications of Theorems A, B and D. We say that \mathcal{F} satisfies the p -boundary condition if \mathcal{F} satisfies the $\{p\}$ -boundary condition in the class of all groups.

Lemma 4.1. *Let $\mathcal{F} = LF(F)$, where F the canonical local satellite of \mathcal{F} . Suppose that for some prime p we have $F(p) = \mathcal{F}$. Then \mathcal{F} does not satisfy the p -boundary condition.*

Proof. Indeed, in this case every \mathcal{F} -critical group is also $F(p)$ -critical.

A group G is called π -closed if G has a normal Hall π -subgroup.

Proposition 4.2. *The formation \mathcal{F} of all π -closed groups satisfies the π' -boundary condition,*

but \mathcal{F} does not satisfy the p -boundary condition for any $p \in \pi$.

Proof. Let $\mathcal{F} = \mathcal{G}_\pi \mathcal{G}_{\pi'}$ be the formation of all π -closed groups, F the canonical local satellite of \mathcal{F} . Then $F(p) = \mathcal{F}$ for all $p \in \pi$, and $F(p) = \mathcal{G}_{\pi'}$ for all primes $p \in \pi'$ by Theorem 3.1.20 in [6]. Hence \mathcal{F} satisfies the π' -boundary condition and does not satisfy the p -boundary condition for any $p \in \pi$ by Lemma 4.1.

Lemma 4.3. *Let $\{\mathcal{F}_i \mid i \in I\}$ be any set of non-empty saturated formations and $\mathcal{F} = \cap_{i \in I} \mathcal{F}_i$.*

(1) *If for each $i \in I$, \mathcal{F}_i satisfies the p -boundary condition, then \mathcal{F} satisfies the p -boundary condition.*

(2) *Suppose that $I = \{1, 2\}$, F_i is the canonical local satellite of \mathcal{F}_i and that there is a set π of primes satisfying the following conditions:*

(a) *\mathcal{F}_1 satisfies the π -boundary condition, and for any $p \in \pi$, we have $F_1(p) \subseteq \mathcal{F}_2 = F_2(p)$ and every $F_1(p)$ -critical group belongs to \mathcal{F}_2 .*

(b) *\mathcal{F}_2 satisfies the π' -boundary condition, and for any $p \in \pi'$, we have $F_2(p) \subseteq \mathcal{F}_1 = F_1(p)$ and every $F_2(p)$ -critical group belongs to \mathcal{F}_1 .*

Then \mathcal{F} satisfies the boundary condition.

Proof. (1) Let F_i be the canonical local satellite of \mathcal{F}_i and F the canonical local satellite of \mathcal{F} . If $f(p) = \cap_{i \in I} F_i(p)$, then $F(p) = \mathcal{G}_p f(p)$ by Theorem 3.3 in [10, Chapter 1]. Now let G be any $F(p)$ -critical group, $i \in I$. Since $F(p) \subseteq F_i(p)$, all maximal subgroup of G belongs to $F_i(p)$. Hence $G \in \mathcal{F}_i$ since $F_i(p) \subseteq \mathcal{F}_i$ and \mathcal{F}_i satisfies the p -boundary condition. This implies that $G \in \mathcal{F}$ and therefore \mathcal{F} satisfies the p -boundary condition.

(2) In this case, $F(p) = F_1(p)$ for all $p \in \pi$ and $F(p) = F_2(p)$ for all $p \in \pi'$, where F is the canonical local satellite of \mathcal{F} . Hence if $p \in \pi$ and G is an $F(p)$ -critical groups, then $G \in \mathcal{F}$ by hypothesis (a). This shows that \mathcal{F} satisfies the π -boundary condition. Similarly we see that \mathcal{F} satisfies the π' -boundary condition.

A group G is called p -decomposable if there exists a subgroup H of G such that $G = P \times H$ for some (and hence the unique) Sylow p -subgroup P of G .

Corollary 4.4. *The formation of all p -decomposable groups satisfies the boundary condition.*

Proof. Let \mathcal{F} be the formation of all p -decomposable groups. Then $\mathcal{F} = \mathcal{G}_{p'} \mathcal{G}_p \cap \mathcal{G}_p \mathcal{G}_{p'}$. Hence the assertion follows from Proposition 4.2 and Lemma 4.3(2).

From Corollary 4.4 and Theorem A we get

Corollary 4.5. *Let D be the intersection of all maximal p -decomposable subgroups of G . Then D is the largest normal subgroup of G satisfying $D = O_{p'}(D) \times O_p(D)$, and G induces the trivial automorphisms group on every chief factor of G below $O_p(D)$ and a π' -group of automorphisms on every chief factor of G below $O_{p'}(D)$.*

Since a p -nilpotent group is p' -closed, the following result directly follows from Proposition 4.2.

Corollary 4.6. *The formation of all p -nilpotent groups satisfies the p -boundary condition.*

From Corollary 4.6 and Theorem A, we have

Corollary 4.7. *Let D be the intersection of all maximal p -nilpotent subgroups of a group G . Then D is the largest normal subgroup of G satisfying $O_{p'}(D) = O_{p'}(G)$, and $D/O_{p'}(G) \leq Z_\infty(G/O_{p'}(G))$.*

Note that another proof of Corollary 4.7 was obtained in the paper [3].

Remark 4.8. If $\pi \neq \{2\}$, then the formation \mathcal{F} of all π -supersoluble groups does not satisfy the π -boundary condition. Indeed, let F be the canonical local satellite of \mathcal{F} . Then $F(p) = \mathcal{N}_p\mathcal{A}(p-1)$, where $\mathcal{A}(p-1)$ is the formation of all abelian groups of exponent dividing $p-1$ [4, p.358], for all $p \in \pi$ and $F(p) = \mathcal{F}$ for all primes $q \notin \pi$ (see Example 3.4(e) and Theorem 4.8 in [4, Chapter IV]). Let $2 \neq p \in \pi$ and q any prime with q divides $p-1$. Let $G = Q \rtimes C$, where C is a group of order p and Q is an $\mathbb{F}_q C$ -module which is faithful for C . Then G is a non-supersoluble $F(p)$ -critical groups.

Proposition 4.9. *Let $\mathcal{F} = \mathcal{NL}$.*

(i) *If \mathcal{L} is a hereditary saturated formation satisfying the boundary condition in the class of all soluble groups, then \mathcal{F} satisfies the boundary condition in the class of all soluble groups.*

(ii) *If \mathcal{L} is a formation of nilpotent groups with $\pi(\mathcal{L}) = \mathbb{P}$, then \mathcal{F} satisfies the boundary condition.*

Proof. Let F be the canonical local satellite of \mathcal{F} . Then by Lemma 2.9(2), $F(p) = \mathcal{G}_p\mathcal{L}$ for all primes p . Assume that \mathcal{F} does not satisfy the boundary condition (\mathcal{F} does not satisfy the boundary condition in the class of all soluble groups). Then for some prime p , the set of all $F(p)$ -critical (soluble) groups A with $A \notin \mathcal{F}$ is non-empty. Let G be a group of minimal order in this set. Then $L = G^\mathcal{F}$ is the unique minimal normal subgroup of G and $O_p(G) = 1 = \Phi(G)$ by Lemma 2.6. First suppose that G is soluble. Then $L = C_G(L)$ is a q -group for some prime $q \neq p$ and $G = L \rtimes M$ for some maximal subgroup M of G with $O_q(M) = 1$ by [4, Chapter A, Theorem 15.6]. Let M_1 be any maximal subgroup of M . Then $LM_1 \in F(p)$, so $LM_1 \in \mathcal{H}$ since $L = C_G(L)$.

(i) Let L be the canonical local satellite of \mathcal{L} . Since $LM_1 \in \mathcal{L}$,

$$LM_1/O_{q',q}(LM_1) = LM_1/O_q(LM_1) = LM_1/LO_q(M_1) \simeq M_1/M_1 \cap LO_q(M_1) \in L(q) = \mathcal{G}_pL(q).$$

Hence every maximal subgroup of M belongs to $L(q)$. Since \mathcal{L} satisfies the boundary condition in the class of all soluble groups, it follows that $G \in \mathcal{F}$. This contradiction completes the poof of (i).

(ii) We may suppose that M_1 is a normal maximal subgroup of M . Then $LM_1 \in F(p)$, so $LM_1 \in \mathcal{H}$ since $L = C_G(L)$. This implies that $M_1 = 1$. Consequently $|M|$ is prime, so $M \in \mathcal{L}$ since $\pi(\mathcal{L}) = \mathbb{P}$. But then $G \in \mathcal{F} = \mathcal{NL}$. Therefore G is not soluble. Let $q \neq p$ be any prime divisor of $|G|$. Suppose that G is not q -nilpotent. Then G has a q -closed \mathcal{N} -critical subgroup $H = Q \rtimes R$ by [7, IV, Theorem 5.4], where Q is a Sylow q -subgroup of H , R is a cyclic Sylow r -subgroup of H . Since G is not soluble, $H \neq G$. Hence $H \leq M \in F(p)$ for some maximal subgroup M of G . Since $M \in \mathcal{G}_p\mathcal{L}$, $M^\mathcal{N} \leq O_p(M)$ and hence $H^\mathcal{N} \leq Q \cap O_p(H) = 1$. Therefore H is nilpotent. This contradiction shows

that G is q -nilpotent for all primes $q \neq p$. This induces that $G^{\mathcal{N}}$ is a Sylow p -subgroup of G and thereby G is soluble. This contradiction completes the poof of (ii).

Remark 4.10. The condition " \mathcal{L} is a hereditary saturated formation satisfying the boundary condition in the class of all soluble groups" can not be omitted in the Statement (ii). Indeed, let $\mathcal{F} = \mathcal{N}\mathcal{U}$ and $G = P \rtimes A_4$, where P is a simple $\mathbb{F}_3 A_4$ -module P which is faithful for A_4 . Let F be the canonical local satellite of \mathcal{F} . Then $F(2) = \mathcal{F}_2\mathcal{U}$ by Lemma 2.9 (2). Therefore G is an $F(2)$ -critical group and $G \notin \mathcal{F}$. Thus G does not satisfy the boundary condition in the class of all soluble groups.

Corollary 4.11. *Let \mathcal{F} be the class of all groups with $G' \leq F(G)$. Then \mathcal{F} satisfies the boundary condition.*

From Corollary 4.11 and Theorem A, we obtain

Corollary 4.12. *Let D be the intersection of all maximal subgroups H of G with the property $H' \leq F(H)$. Then D is the largest normal subgroup of G such that $D' \leq F(D)$ and G induces an abelian group of automorphisms on every chief factor of G below D .*

Note that Corollary 4.12 can be also found in [3, Corollary 7 and Remark 4].

Following [4, Chapter VII, Definitions 6.9] we write $l(G)$ to denote the nilpotent length of the group G . Recall that \mathcal{N}^r is the product of r copies of \mathcal{N} ; \mathcal{N}^0 is the class of groups of order 1 by definition. It is well known that \mathcal{N}^r is the class of all soluble groups G with $l(G) \leq r$. It is also known that \mathcal{N}^r is a hereditary saturated formation (see, for example, [4, p. 358]). Hence from Proposition 4.9 we get.

Corollary 4.13. *Let $\mathcal{F} = \mathcal{N}^r \mathcal{L}$ ($r \in \mathbb{N}$), where \mathcal{L} is a subformation of the formation of all abelian groups with $\pi(L) = \mathbb{P}$. Then \mathcal{F} satisfies the boundary condition in the class of all soluble groups.*

From Proposition 4.9 and Theorem A we get

Corollary 4.14. *Let D be the intersection of all maximal metanilpotent subgroups of a group G . Then D is the largest normal subgroup of G satisfying D is metanilpotent and G induces a nilpotent group of automorphisms on every chief factor of G below D .*

It is clear that every subnormal subgroup is a K - \mathcal{F} -subnormal subgroup as well. On the other hand, in the case, where $\mathcal{N} \subseteq \mathcal{F}$, every K - \mathcal{F} -subnormal subgroup of any soluble subgroup G is \mathcal{F} -subnormal in G . Hence from Theorems D and the above corollaries we get

Corollary 4.15. *Let \mathcal{F} be the class of all groups G with $G' \leq F(G)$.*

(i) *The subgroup $Z_{\mathcal{F}}(G)$ may be characterized as the intersection of all non-subnormal \mathcal{F} -maximal subgroups of G , for each group G .*

(ii) *The subgroup $Z_{\mathcal{F}}(G)$ may be characterized as the intersection of all non- \mathcal{F} -subnormal \mathcal{F} -maximal subgroups of G , for each soluble group G .*

Corollary 4.16. *Let \mathcal{F} be one of the following formations:*

(1) *the class of all nilpotent groups (Baer [1]);*

(2) the class of all groups G with $G' \leq F(G)$ (Skiba [13]);

(3) the class of all p -decomposable groups (Skiba [12]).

Then for each group G , the equality $\text{Int}_{\mathcal{F}}(G) = Z_{\mathcal{F}}(G)$ holds.

Corollary 4.17 (Sidorov [9]). Let \mathcal{F} be the class of all soluble groups G with $l(G) \leq r$ ($r \in \mathbb{N}$). Then for each soluble group G , the equality $Z_{\mathcal{F}}(G) = \text{Int}_{\mathcal{F}}(G)$ holds.

An application of Theorem C (a). Let $p_1 > p_2 > \dots > p_r$ be the distinct primes dividing $|G|$, P_i a Sylow p_i -subgroup of G . Then G is said to satisfy *the Sylow tower property* if all subgroups $P_1, P_1P_2, \dots, P_1P_2 \dots P_{r-1}$ are normal in G . Every supersoluble group satisfy the Sylow tower property.

Proposition 4.18. Let \mathcal{F} be the formation of all supersoluble groups and \mathcal{M} be the formation of all groups satisfying the Sylow tower property. Then $\text{Int}_{\mathcal{F}}(G) \leq \text{Int}_{\mathcal{M}}(G)$ for any group G .

Proof. Let F be the canonical local satellite of \mathcal{F} . Then $F(p) = \mathcal{G}_p\mathcal{A}(p-1)$ for primes p (see Remark 4.8). Let G be any $F(p)$ -critical group satisfying the Sylow tower property. We show that $G \in \mathcal{F}$. Let q be the largest prime dividing $|G|$, P the Sylow q -subgroup of G . If $G = P$, then clearly $G \in \mathcal{F}$. Let $P \neq G$. Then every Sylow subgroup of G belongs to $F(p)$. Hence $q = p$ and if E is a Hall p' -subgroup of G , then $E \in \mathcal{A}(p-1)$. But then $G \in F(p)$, a contradiction. Therefore $G \in \mathcal{F}$. This shows that \mathcal{F} satisfies the boundary condition in \mathcal{M} . Hence, in view of Theorem C(b), we have $\text{Int}_{\mathcal{F}}(G) \leq \text{Int}_{\mathcal{M}}(G)$.

Remark 4.19. If $\mathcal{F} \subseteq \mathcal{M}$ be hereditary saturated formations, then for every group G we have $Z_{\mathcal{F}}(G) \leq Z_{\mathcal{M}}(G)$ and $Z_{\mathcal{F}}(G) \leq \text{Int}_{\mathcal{F}}(G)$ by Lemma 2.4(g). Therefore, if \mathcal{F} satisfies the boundary condition, then $\text{Int}_{\mathcal{F}}(G) \leq \text{Int}_{\mathcal{M}}(G)$ for every group G . But, by Remark 4.8, \mathcal{U} does not necessarily satisfy the boundary condition. Hence we can not deduce Proposition 4.18 from Theorem A.

Lemma 4.20. Let $\mathcal{F} = LF(F)$ be a saturated formations, where F is the canonical local satellites of \mathcal{F} . Let $\mathcal{M} = \mathcal{F} \cap \mathcal{S}$ and M the canonical local satellites of \mathcal{M} . Then $M(p) = F(p) \cap \mathcal{S}$ for all primes p .

Proof. It is clear that $\mathcal{M} = LF(F_1)$, where $F_1(p) = F(p) \cap \mathcal{S}$ for all primes p . On the other hand, for any p we have $F_1(p) \subseteq \mathcal{M}$, and $F_1(p) = \mathcal{G}_p F_1(p)$ since

$$\mathcal{G}_p F_1(p) = \mathcal{G}_p (F(p) \cap \mathcal{S}) \subseteq F(p) \cap \mathcal{S}.$$

Hence $F_1 = M$ is the canonical local satellite of \mathcal{M} .

Lemma 4.21. Let $(1) \neq \mathcal{F} = LF(F)$ be a saturated formations and $\mathcal{M} = \mathcal{F} \cap \mathcal{S}$. Let $\pi \subseteq \pi(\mathcal{F})$. Then \mathcal{F} satisfies the boundary condition in the class \mathcal{S} of all soluble groups if and only if \mathcal{M} satisfies the boundary condition in \mathcal{S} .

Proof. Let F and M be the canonical local satellite of the formation \mathcal{F} and \mathcal{M} , respectively. Let $p \in \pi$. Then $M(p) = F(p) \cap \mathcal{S}$ by Lemma 4.20. Hence a soluble group G is $F(p)$ -critical group if and only if G is $M(p)$ -critical. On the other hand, the group G belongs to \mathcal{F} if and only if G belongs to \mathcal{M} .

In view of Lemmas 4.20 and 4.21, Theorem B may be proved in the following general form.

Theorem 4.22. *Let \mathcal{F} be a saturated formation such that $\mathcal{M} = \mathcal{F} \cap \mathcal{S}$ is hereditary and $(1) \neq \mathcal{M} \neq \mathcal{S}$. Let $\pi \subseteq \pi(\mathcal{F})$. Then the equality*

$$Z_{\pi\mathcal{F}}(G) = \text{Int}_{\mathcal{F}}(G)$$

holds for each soluble group G if and only if $\mathcal{N} \subseteq \mathcal{F} = \mathcal{S}_{\pi'}\mathcal{F}$ and \mathcal{F} satisfies the boundary condition in the class of all soluble groups.

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